EULER-CAUCHY NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

[higher order] $[b_0 x^n D^n + b_1 x^{n-1} D^{n-1} + \ldots + b_{n-1} xD + b_n]y = h(x)$

can be expressed in terms of constant coefficients by changing independent variable x to z with $x = e^z$.

$$\because x = e^z \quad \therefore z = \ln x \quad \textbf{and} \quad \because \frac{dx}{dz} = e^z \quad \therefore \frac{dz}{dx} = e^{-z} = \frac{1}{x}$$

Defintion of differential operator : $D_z = \dfrac{d}{dz},\ D_z^2 = \dfrac{d^2}{dz^2}, \cdots, D_z^2 = \dfrac{d^n}{dz^n}$

$$\because Dy = \frac{dy}{dx} = \frac{dz}{dx}\frac{dy}{dz} = e^{-z}\frac{dy}{dz} = e^{-z} D_z y = \frac{1}{x} D_z y \qquad \therefore xDy = D_z y$$

$$\because \begin{cases} D^2 y = \dfrac{d^2 y}{dx^2} = \dfrac{d}{dx}[e^{-z} D_z y] = \dfrac{dz}{dx}\dfrac{d}{dz}[e^{-z} D_z y] = e^{-z}[e^{-z} D_z^2 y - e^{-z} D_z y] = e^{-2z} D_z (D_z - 1) y \\ \quad = \dfrac{1}{x^2} D_z (D_z - 1) y \end{cases} \qquad \therefore x^2 D^2 y = D_z (D_z - 1) y$$

Similarly, $x^3 D^3 y = D_z (D_z - 1)(D_z - 2) y, \cdots, \ x^n D^n y = D_z (D_z - 1)(D_z - 2)\ldots[D_z - (n-1)] y$

Substituting to the equation gives an ***n*-order linear differential equation with constant coefficient**

$$\left[b_0 D_z (D_z - 1)(D_z - 2)\ldots[D_z - (n-1)] + b_1 D_z (D_z - 1)(D_z - 2)\ldots[D_z - (n-2)] + \cdots + b_{n-2} D_z (D_z - 1) + b_{n-1} D_z + b_n\right] y = h(e^z)$$

$$x^2y'' - 4xy' + 6y = 6x + 12$$

Let $x = e^z \Rightarrow z = \ln x$ **and** $[D_z(D_z - 1) - 4D_z + 6]y = 6e^z + 12$

$D_z(D_z - 1) - 4D_z + 6 = 0 \Rightarrow D_z = 2,\ 3$ **and** $y_h = c_1 e^{2z} + c_2 e^{3z} = c_1 x^2 + c_2 x^3$

(1) $y_p = \dfrac{6}{(D_z - 2)(D_z - 3)} e^z + \dfrac{1}{D_z^2 - 5D_z + 6} 12 = \dfrac{6}{(-1)(-2)} e^z + (\dfrac{1}{6} + \cdots)12 = 3e^z + 2 = 3x + 2$

(2) $y_p = u(z)e^{2z} + v(z)e^{3z}$

$$u(z) = \int \frac{\begin{vmatrix} 0 & e^{3z} \\ 6e^z + 12 & 3e^{3z} \end{vmatrix}}{\begin{vmatrix} e^{2z} & e^{3z} \\ 2e^{2z} & 3e^{3z} \end{vmatrix}} dz = -\int \frac{6e^{4z} + 12e^{3z}}{e^{5z}} dz = -\int (6e^{-z} + 12e^{-2z}) dz = 6e^{-z} + 6e^{-2z}$$

$$v(z) = \int \frac{\begin{vmatrix} e^{2z} & 0 \\ 2e^{2z} & 6e^z + 12 \end{vmatrix}}{e^{5z}} dz = \int \frac{6e^{3z} + 12e^{2z}}{e^{5z}} dz = \int (6e^{-2z} + 12e^{-3z}) dz = -3e^{-2z} - 4e^{-3z}$$

$$y_p = (6e^{-z} + 6e^{-2z})e^{2z} + (-3e^{-2z} - 4e^{-3z})e^{3z} = 6e^z + 6 - 3e^z - 4 = 3e^z + 2 = 3x + 2$$

G.S. $y = c_1 x^2 + c_2 x^3 + 3x + 2$

LEGENDRE DIFFERENTIAL EQUATION

[Higher order]

$$b_0(ax + b)^n \frac{d^n y}{dx^n} + b_1(ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + b_{n-1}(ax + b)\frac{dy}{dx} + b_n y = h(x)$$

$$[b_0(ax - b)^n D^n + b_1(ax + b)^{n-1} D^{n-1} + \ldots + b_{n-1}(ax + b)D + b_n]y = h(x)$$

$$(ax+b)=e^{z} \Rightarrow \begin{cases} z=\ln(ax+b) \\ \dfrac{dz}{dx}=\dfrac{a}{ax+b}=ae^{-z} \end{cases}$$ **and Defintion of differential operator :** $D_z=\dfrac{d}{dz},\ D_z^2=\dfrac{d^2}{dz^2},\cdots,D_z^n=\dfrac{d^n}{dz^n}$

$$\left\{ \because Dy=\frac{dy}{dx}=\frac{dz}{dx}\frac{dy}{dz}=ae^{-z}\frac{dy}{dz}=ae^{-z}D_z y=\frac{a}{ax+b}D_z y \quad \therefore\ (ax+b)Dy=aD_z y \right.$$

$$\begin{cases} \because D^2y=\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}[ae^{-z}D_z y]=\dfrac{dz}{dx}\dfrac{d}{dz}[ae^{-z}D_z y]=ae^{-z}[ae^{-z}D_z^2 y-ae^{-z}D_z y] \\ \qquad =a^2e^{-2z}D_z(D_z-1)y=\dfrac{a^2}{(ax+b)^2}D_z(D_z-1)y \quad \therefore (ax+b)^2D^2y=a^2D_z(D_z-1)y \end{cases}$$

Similarly, $(ax+b)^3D^3=a^3D_z(D_z-1)(D_z-2);\ (ax+b)^4D^4=a^3D_z(D_z-1)(D_z-2)(D_z-3);\cdots\cdots;$

$$(ax+b)^nD^n=a^nD_z(D_z-1)\ldots[D_z-(n-1)]$$

Example $(3x+2)^2\ y''+3(3x+2)\ y'-36\ y=3x^2+4x+1$

Let $3x+2=e^z \Rightarrow z=\ln(3x+2)$

$$\because (3x+2)^2=(9x^2+12x+4)=3(3x^2+4x+\frac{4}{3}) \quad \therefore 3x^2+4x+1=\frac{1}{3}(3x+2)^2-\frac{1}{3}$$

$$[9D_z(D_z-1)+9D_z-36]\ y=9(D_z^2-2)y=\frac{1}{3}(e^{2z}-1)\Rightarrow(D_z^2-4)\ y=\frac{1}{27}(e^{2z}-1)$$

$$\Rightarrow \begin{cases} y_h=c_1e^{2z}-c_2e^{-2z} \\ y_p=\dfrac{1}{27}\dfrac{1}{D_z^2-4}(e^{2z}-1)=\dfrac{1}{27}[\dfrac{1}{(D_z-2)(D_z+2)}e^{2z}-\dfrac{1}{D_z^2-4}1]=\dfrac{1}{27}(\dfrac{ze^{2z}}{4}+\dfrac{1}{4})=\dfrac{1}{108}(ze^{2z}+1) \end{cases}$$

$$\therefore y(z)=c_1e^{2z}-c_2e^{-2z}+\frac{1}{108}(ze^{2z}+1) \quad \therefore \textbf{G.S.} \quad y(x)=c_1(3x+2)^2+c_2(3x+2)^{-2}+\frac{1}{108}[(3x+2)^2\ln(3x+2)^2+1]$$

[Theorem 1] For a given function $f(t)$ that is defined for all $t \geq 0$, the Laplace transform $£[f(t)]$, provided the integral converges, is defined by multiplying $f(t)$ with e^{-st} and integrating with respect to t from zero to infinity, i.e.

$$£[f(t)] = \int_0^\infty f(t)e^{-st}dt = F(s).$$

The integration variable is t as the variable of original function, and hence the integral defines a function of the new variable s as the variable of its Laplace Transform.

The inverse Laplace Transform of $F(s)$ is a function $f(t)$, meaning that $£[f(t)] = F(s)$ implies $£^{-1}[F(s)] = f(t)$.

[Theorem 2] Summary
Unit step function $u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$ **gives** $£[1] = \frac{1}{s}$, $£[N] = \frac{N}{s}$ **and** $£^{-1}[\frac{1}{s}] = 1$, $£^{-1}[\frac{N}{s}] = N$ **for** $s > 0$.
Power function t^n **gives** $£[f(t)] = £[t^n] = \frac{n!}{s^{n+1}}$ **and** $£^{-1}(\frac{n!}{s^{n+1}}) = t^n$ **or** $£^{-1}(\frac{1}{s^{n+1}}) = \frac{t^n}{n!}$, **for** $s > 0$.
Exponential functions e^{-at} **and** e^{at} **when** $t \geq 0$, **where** a **is constant, give** $\begin{cases} £[e^{at}] = \frac{1}{s-a}, \\ £[e^{-at}] = \frac{1}{s+a} \end{cases} \Leftrightarrow \begin{cases} £^{-1}[\frac{1}{s-a}] = e^{at}, \\ £^{-1}[\frac{1}{s+a}] = e^{-at} \end{cases}$ **for** $s > a$.
$\cos(at)$ **and** $\cosh(at)$, **when** $t \geq 0$, **where** a **is constant, give** $\begin{cases} £[\cos(at)] = \frac{s}{s^2+a^2} \\ £[\cosh(at)] = \frac{s}{s^2-a^2} \end{cases} \Leftrightarrow \begin{cases} £^{-1}(\frac{s}{s^2+a^2}) = \cos(at) \\ £^{-1}(\frac{s}{s^2-a^2}) = \cosh(at) \end{cases}$, $s > a\ (\geq 0)$.
$\sin(at)$ **and** $\sinh(at)$, **when** $t \geq 0$, **where** a **is constant, give** $\begin{cases} £[\sin(at)] = \frac{a}{s^2+a^2} \\ £[\sinh h(at)] = \frac{a}{s^2-a^2} \end{cases} \Leftrightarrow \begin{cases} £^{-1}(\frac{a}{s^2+a^2}) = \sin(at) \\ £^{-1}(\frac{a}{s^2-a^2}) = \sinh(at) \end{cases}$, $s > a\ (\geq 0)$.

[Theorem 3] Laplace Transform of Derivative

The Laplace Transforms of differentiation of $f(t)$ **is replaced by multiplication of** $£[f(t)]$ **by** s **such as**

$$£[f^{(n)}(t)] = s^n\, £[f(t)] - s^{n-1}\, f(0) - s^{n-2}\, f'(0)\ldots f^{n-1}(0) \quad \textbf{for} \quad s > \alpha.$$

$$£[f'(t)] = s\,£[f(t)] - f(0)$$

$$£[f''(t)] = s^2\, £[f(t)] - s\, f(0) - f'(0)$$

[Theorem 4] Function Defined By Integral

The Laplace Transforms of integration of $f(t)$ **is replaced by division of** $£[f(t)]$ **by** s **such as**

$$£[\int_0^t\int_0^t\int_0^t\ldots\ldots\int_0^t f(t)\, dt^{(n)}] = \frac{1}{s^n}\, £[f(t)] \quad \textbf{and} \quad £^{-1}\{\frac{1}{s^n}\, £[f(t)]\} = \int_0^t\int_0^t\int_0^t\ldots\ldots\int_0^t f(t)\, dt^{(n)}.$$

$y' + y + 3\int_0^t z\, dt = \cos t + 3\sin t$
$2y' + 3z' + 6z = 0$
. **Find** $y(t)$ **at** $y(0) = -3,\ z(0) = 2$.

$$\begin{cases} £[y'] + £[y] + 3\,£[\int_0^t z\, dt] = £[\cos t] + 3\,£[\sin t] \Rightarrow s\, £[y] - y(0) + £[y] + \frac{3}{s}£[z] = \frac{s}{s^2+1} + \frac{3}{s^2+1} \\ 2\,£[y'] + 3\,£[z'] + 6\,£[z] = 0 \Rightarrow 2s\,£[y] - 2y(0) + 3s\,£[z] - 3z(0) + 6\,£[z] = 0 \end{cases} \Rightarrow \begin{cases} (s+1)\,£[y] + \frac{3}{s}£[z] = \frac{s+3}{s^2+1} - 3 \\ 2s\,£[y] + 3(s+2)\,£[z] = 0 \end{cases}$$

$$£[y] = \frac{\begin{vmatrix} [\frac{s+3}{s^2+1} - 3] & \frac{3}{s} \\ 0 & 3(s+2) \end{vmatrix}}{\begin{vmatrix} s+1 & \frac{3}{s} \\ 2s & 3(s+2) \end{vmatrix}} = \frac{(\frac{s+3}{s^2+1} - 3)\, 3\,(s+2)}{(s+1)\, 3\,(s+2) - 6} = \frac{3(s+3)(s+2) - 9(s+2)(s^2+1)}{3s(s+3)(s^2+1)} = \frac{s+2}{s(s^2+1)} - 3\frac{s+2}{s(s+3)}$$

$$\begin{cases} £^{-1}[\frac{s+2}{s(s^2+1)}] = £^{-1}\{\frac{1}{s}\,£[\cos(t)] + 2\frac{1}{s}\,£[\sin(t)]\} = \int_0^t \cos(t)dt + 2\int_0^t \sin(t)dt = \sin(t) - 2\cos(t) + 2 \\ 3\,£^{-1}[\frac{s+2}{s(s+3)}] = 3\,£^{-1}[\frac{1/3}{s+3} + \frac{2/3}{s}] = £^{-1}[\frac{1}{s+3} + \frac{2}{s}] = e^{-3t} + 2 \end{cases} \Rightarrow y(t) = -2\cos(t) + \sin(t) - e^{-3t}, \quad t > 0.$$

FURTHER PROPERTIES OF LAPLACE TRANSFORM

[Theorem A] First Shifting theorem (Shifting in the S-Variable) 第一移位定理

$$\begin{cases} £[e^{at} f(t)] = £[f(t)]_{s\to s-a} = F(s-a) \Leftrightarrow £^{-1}[F(s-a)] = e^{at}\, £^{-1}[F(s)] = e^{at} f(t) \quad \textbf{for} \quad s-a>0 \\ £[e^{-at} f(t)] = £[f(t)]_{s\to s+a} = F(s+a) \Leftrightarrow £^{-1}[F(s+a)] = e^{-at}\, £^{-1}[F(s)] = e^{-at} f(t) \quad \textbf{for} \quad s+a>0 \end{cases}$$

Use Laplace Transform to solve an initial value problem, $y''(t)+2y'(t)+5y(t)=e^{-t}\sin(t);\quad y(0)=0,\ y'(0)=1$.

Solution $\quad (s^2+2s+5)Y(s)-1=\dfrac{1}{(s+1)^2+1} \Rightarrow (s^2+2s+5)Y(s)=\dfrac{s^2+2s+3}{(s+1)^2+1}$

$$\because Y(s)=\frac{s^2+2s+3}{(s^2+2s+5)[(s+1)^2+1]}=\frac{(s+1)^2+2}{[(s+1)^2+4][(s+1)^2+1]}=\frac{1}{3}[\frac{2}{(s+1)^2+4}+\frac{1}{(s+1)^2+1}]$$

$$\therefore y(t)=\frac{1}{3}e^{-t}[\sin(2t)+\sin(t)]$$

[Theorem B] Differentiation of Laplace Transform

$$F(s)=£[f(t)] \Rightarrow \frac{d^nF(s)}{ds^n}=(-1)^n\, £[t^n f(t)] \quad \textbf{or} \quad £[t^n f(t)]=(-1)^n\frac{d^n}{ds^n}F(s)$$

Solve the initial value problem $ty''+(4t-2)y'-4y=0;\qquad y(0)=1$.

Solution **Let** $\pounds[y(t)] = Y(s) \Rightarrow (s^2+4s)Y'(s)+(4s+8)Y(s)=3$ **or** $Y'(s)+\dfrac{4s+8}{s(s+4)}Y(s)=\dfrac{3}{s(s+4)}$

$$Y(s)=e^{-\int\frac{4s+8}{s(s+4)}ds}\left[\int e^{\int\frac{4s+8}{s(s+4)}ds}\frac{3}{s(s+4)}ds+c\right]\Rightarrow\begin{cases} e^{-\int\frac{2(2s+4)}{s^2+4s}ds}=e^{-2\int\frac{1}{s^2+4s}d(s^2+4s)}=e^{-2\ln(s^2+4s)}=\dfrac{1}{s^2(s+4)^2} \\ e^{\int\frac{2(2s+4)}{s^2+4s}ds}=e^{2\int\frac{1}{s^2+4s}d(s^2+4s)}=e^{2\ln(s^2+4s)}=s^2(s+4)^2\end{cases}$$ **and** $$\int e^{\int\frac{4s+8}{s(s+4)}ds}\frac{3}{s(s+4)}ds=s^2(s+6)$$

$$Y(s)=\frac{s^2(s+6)}{s^2(s+4)^2}+\frac{c}{s^2(s+4)^2}\qquad\begin{cases}\pounds^{-1}[\dfrac{s+4+2}{(s+4)^2}]=\pounds^{-1}[\dfrac{s+4}{(s+4)^2}+\dfrac{2}{(s+4)^2}]=e^{-4t}\pounds^{-1}[\dfrac{1}{s}+\dfrac{2}{s^2}]=e^{-4t}(1+2t)\\ \pounds^{-1}[\dfrac{c}{s^2(s+4)^2}]=c\int_0^t\int_0^t te^{-4t}(dt)^2=c[-\dfrac{1}{32}+\dfrac{1}{16}t+\dfrac{1}{16}te^{-4t}+\dfrac{1}{32}e^{-4t}]\end{cases}$$

G.S. $y(t)=e^{-4t}+2te^{-4t}-c\,[\dfrac{1}{32}-\dfrac{1}{16}t-\dfrac{1}{16}te^{-4t}-\dfrac{1}{32}e^{-4t}]$

[Theorem C] Integration of Laplace Transform

$$\boxed{\pounds[f(t)]=F(s)\Rightarrow\int_s^\infty\cdots\int_s^\infty F(s)(ds)^n=\pounds[\frac{f(t)}{t^n}]\ \textbf{ or }\ \pounds[\frac{f(t)}{t^n}]=\int_s^\infty\ldots\int_s^\infty F(s)\,(ds)^n}$$

[Theorem D] Second Shifting theorem (in the t-Variable) 第二移位定理

$$\text{IF }\pounds[f(t)]=F(s),\text{ then }\begin{cases}\pounds[f(t-a)\,u(t-a)]=e^{-as}\,\pounds[f(t)]=e^{-as}F(s)\\ \pounds^{-1}[e^{-as}F(s)]=u(t-a)\{\pounds^{-1}[F(s)]\}_{t\to t-a}\end{cases}$$

Solve the initial value problem $y''+4y=f(t)$, $y(0)=y'(0)=0$ **and** $f(t)=\begin{cases}0 & t<3\\ t & t\ge 3\end{cases}$.

Let $\pounds[y]=Y(s),\ \pounds[y''+4y=f(t)]\Rightarrow s^2Y-sy(0)-y'(0)+4Y=\pounds[f(t)]$

$\because \pounds[f(t)]=\pounds[tu(t-3)]=\pounds\{[(t-3)+3]u(t-3)\}=\pounds[(t-3)u(t-3)]+\pounds[3u(t-3)]=\frac{1}{s^2}e^{-3s}+\frac{3}{s}e^{-3s}$

$\therefore (s^2+4)Y=\frac{3s+1}{s^2}e^{-3s}\ \Rightarrow\ Y=\frac{3s+1}{s^2(s^2+4)}e^{-3s}=\frac{1}{s^2}[\frac{3s}{s^2+4}e^{-3s}+\frac{1}{s^2+4}e^{-3s}]\Rightarrow y(t)=\pounds^{-1}\{\frac{1}{s^2}\pounds[3\cos 2(t-3)u(t-3)+\frac{1}{2}\sin 2(t-3)u(t-3)]\}$

For $t\ge 3,\quad y(t)=\int_3^t\left(\int_3^t\{3\cos[2(t-3)]+\frac{1}{2}\sin[2(t-3)]\}dt\right)dt$

Let $t-3=\tau,\ y(t)=\int_3^t\left\{\int_0^{t-3}[3\cos(2\tau)+\frac{1}{2}\sin(2\tau)]d\tau\right\}dt=\int_3^t[\frac{3}{2}\sin(2\tau)-\frac{1}{4}\cos(2\tau)]_0^{t-3}dt=\int_3^t\{\frac{3}{2}\sin[2(t-3)]-\frac{1}{4}\cos[2(t-3)]+\frac{1}{4}\}dt$

Let $t-3=\tau$ **again,** $y(t)=\int_0^{t-3}[\frac{3}{2}\sin(2\tau)-\frac{1}{4}\cos(2\tau)+\frac{1}{4}]d\tau=[-\frac{3}{4}\cos(2\tau)-\frac{1}{8}\sin(2\tau)+\frac{1}{4}\tau]_0^{t-3}=-\frac{3}{4}\cos[2(t-3)]-\frac{1}{8}\sin[2(t-3)]+\frac{1}{4}(t-3)+\frac{3}{4}$

$$y(t)=\begin{cases}0 & t<3\\ \frac{1}{8}\{2t-6\cos[2(t-3)]-\sin[2(t-3)]\} & t\ge 3\end{cases}$$

[Theorem E] Reduction by Partial Fraction

(A) $\pounds[f(t)]=\frac{P(s)}{Q(s)}=\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+\frac{A_3}{s-a_3}+\ldots+\frac{A_r}{s-a_r}+\ldots+\frac{A_n}{s-a_n}$

$$f(t)=f_1(t)=\sum_{r=1}^{n}\frac{P(a_r)}{Q'(a_r)}e^{a_rt}$$

(B) $\pounds[f(t)]=\frac{P(s)}{Q(s)}=\frac{P(s)/q(s)}{(s-a)^r}=\frac{\phi(s)}{(s-a)^r}$

$$f(t)=f_2(t)=e^{at}\sum_{n=1}^{r}\frac{\phi^{(r-n)}(a)}{(r-n)!}\frac{t^{n-1}}{(n-1)!}=e^{at}[\frac{\phi^{(r-1)}(a)}{(r-1)!}\frac{t^0}{0!}+\frac{\phi^{(r-2)}(a)}{(r-2)!}\frac{t}{1!}+\frac{\phi^{(r-3)}(a)}{(r-3)!}\frac{t^2}{2!}+\ldots+\frac{\phi'(a)}{1!}\frac{t^{(r-2)}}{(r-2)!}+\phi(a)\frac{t^{(r-1)}}{(r-1)!}]$$

(C) $£[f(t)] = \frac{P(s)}{Q(s)} = \frac{P(s)}{[(s+a)^2+b^2]q(s)} = \frac{\Phi(s)}{[(s+a)^2+b^2]}$

Let $s=-a+ib$, $\Phi(-a+ib) = R_e + i\,I_m$ **and then** $f(t) = f_3(t) = \frac{e^{-at}}{b}[I_m\cos(bt) + R_e\sin(bt)]$

CONVOLUTION THEOREM

It is sometime useful to have a formula for the inverse Laplace transform of a product $F(s)G(s)$ **in terms of the inverse transforms of** $F(s)$ **and** $G(s)$**.**

If $£^{-1}[F(s)] = f(t)$ **and** $£^{-1}[G(s)] = g(t)$**, then**

$$£^{-1}[F(s)G(s)] = £^{-1}\{£[f(t)]\,£[g(t)]\} = \int_0^t f(t-\alpha)\,g(\alpha)\,d\alpha = \int_0^t f(t)\,g(t-\alpha)\,d\alpha\,,$$

where $f(t)*g(t) = \int_0^t f(t-\alpha)\,g(\alpha)\,d\alpha$ **will be a function of** t **and is usually called the convolution of** $f(t)$ **with** $g(t)$**.**

An integral equation is defined as $£[\int_0^t f(t-\alpha)\,g(\alpha)\,d\alpha] = £[f(t)]£[g(t)]$**.**

Solve the initial value problem $y''+4y = 2e^{-t}$**;** $y(0) = y'(0) = 0$**.**

Let $£[y] = Y(s)$

$$s^2\,£[y] - sy(0) - y'(0) + 4\,£[y] = \frac{2}{s+1} \Rightarrow £[y] = \frac{2}{(s+1)(s^2+4)}$$

$$y(t) = £^{-1}\left(£[e^{-t}]\,£[\sin(2t)]\right) = \int_0^t e^{-(t-\alpha)}\sin(2\alpha)d\alpha = \frac{1}{5}[\sin(2t) - 2\cos(2t) + 2e^{-t}]$$

Find a function $f(t)$ **satisfying** $f(t) = e^{-t} + 2\int_0^t e^{-3x} f(t-x)\, dx$

Hint: Integral Equation: $£[\int_0^t f(t-\alpha)\, g(\alpha)\, d\alpha] = £[f(t)]\; £[g(t)]$.

$$£[f(t)] = \frac{1}{s+1} + 2\,£[e^{-3t}]\,£[f(t)] = \frac{1}{s+1} + \frac{2}{s+3}£[f(t)]$$

$$£[f(t)] = \frac{s+3}{(s+1)^2} \text{ and then } f(t) = e^{-t}(1+2t)$$